ON ZEROS OF SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS.

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The subject discussed in this text is the oscillatory behavior of solutions of holomorphic linear time-dependent differential systems of equations. In [1] a bound is provided for the number of times a trajectory of a polynomial vector field crosses a given algebraic hypersurface. This implies a bound also for linear time dependent vector fields with polynomial dependence on time. A different approach is taken here, giving better bound for some special cases.

This text is a preliminary report.

Besides of having an intrinsic interest, a related problem arises in estimating the number of zeros which the so called complete Abelian integrals may have, the latter problem being itself connected to the problem of estimating the number of limit cycles which can be born from polynomial perturbations of Hamiltonian (polynomial) vector fields on the plane (see, for example, [4]).

Remark. The bounds are given here for a real segment in the domain where the coefficients of the equations are holomorphic and real on the real axis, but they can be given in principle for any compact domain contained in the domain where the coefficients are holomorphic ([3]). We also remark that the results on which the bounds appearing here are based, are not the best available (see discussion in [3]).

Let us be given a linear differential system of equations

$$\dot{x} = A(t)x$$

with coefficients holomorphic in some simply connected complex domain. Then the solution will be a vector valued holomorphic function in this domain. The problem in the large is to estimate the number of zeros the first component of the solution may have in that domain.

It is known that if the coefficients of the system belong to the class of polynomials of a given degree, a constructive bound on the number of zeros on some compact subset of the domain may be given which involves just the degree and the maximum modulus of the coefficients [1].

In the following, another way of obtaining bounds on the number of zeros of the solutions of such systems is discussed, which is the consequence of a simple but somewhat surprising fact.

In [1], one of the aims is to bound the complex oscillation of solutions of polynomial vector fields; the result for linear differential systems is obtained as a consequence. One writes the system as a polynomial differential system of equations with the coefficients and time treated as extra variables, the polynomial coefficients being integral. Then, by subsequent differentiating, one obtains the derivatives of the first variable written as polynomials with integer coefficients in all the variables

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of the system. It turns out that one can effectively bound the moment when the ideal generated by the latter polynomials stabilizes. This last derivative is then written as a linear combination of the former derivatives over the ring of polynomials in the variables of the system, which is a linear scalar differential equation for the first coordinate. Since the coefficients of the polynomials are integers, one uses a form of effective Nullstellensatz to bound the coefficients of this equation (on compact regions in the variable space). Since for linear scalar differential equations a bound on the number of zeros exists in terms of the maximum of the coefficients in the region ([2]), a bound on the number of zeros is obtained.

One notes that the obtained scalar differential equation has a space of solutions of dimension much larger than the dimension of the original system, and thus has many solutions not corresponding (at least directly) to the solutions of the system.

In what we do here, we keep the integrality of the polynomial coefficients to be able to use the effective Nullstellensatz. As in [1], we can parameterize in this way all linear differential systems with coefficients being polynomials of a fixed degree. In the beginning, however, we assume only holomorphic dependence on the time and parameters so that not to obscure the main point.

So let there be given a linear differential system of order n $\dot{x} = A(t,p)x = (a_{ij})(t,p)(x)$, where p stands for parameters. We assume the dependence of the system matrix on t, p to be holomorphic in some region U. By W we denote the projection of the domain on the parameter space. We describe a procedure to obtain a scalar differential equation for the first system variable. Differentiating and substituting the relation we have for the first variable, we get:

$$x_1'' = \sum_{i=1}^{n} a_{1i}' x_i + a_{1i} x_i' = \sum_{i=1}^{n} a_{1i}' x_i + a_{1i} \sum_{i=1}^{n} a_{ij} x_j = \sum_{i=1}^{n} (a_{1i}' + \sum_{j=1}^{n} a_{1j} a_{ji}) x_i.$$

Similarly we get the expressions for $x_1^{(i)}$ for any i. We write

$$x_1^{(i)} = \sum a_j^{(i)} x_j = a^{(i)} x.$$

There is a minimal k, $1 \le k \le n$, such that $a^{(0)}, a^{(1)}, .., a^{(k)}$ are linearly dependent. We then have a unique decomposition

$$a^{(k)} = A_{k-1} a^{(k-1)} + ... + A_0 a^{(0)},$$

$A_i \in \mathcal{M}(U)$, implying that the following equation must be satisfied by any solution for the x_1 variable of the initial system

$$x_1^{(k)} - A_{k-1} x_1^{(k-1)} - ... - A_0 x_1 = 0.$$

In the sequel we call this differential equation simply 'the derived equation'. Notice that the determinant of a suitable k-minor of the matrix

$$\left(a^{(0)}, a^{(1)}, .., a^{(k-1)} \right)$$

is not identically zero, and that multiplying by it the equation, we get an equation with holomorphic coefficients. We thus call it the leading coefficient of the derived equation (corresponding to the minor chosen; of course in the case k=n there is only one such choice).

Claim 1. *There exists an analytic set G such that for any $p \in W - G$ any solution of the above equation is a solution for the x_1 variable of the initial system.*

Proof. The leading coefficient of the derived equation is a holomorphic function in p, t, not identically zero. Thus the values of p for which the leading coefficient vanishes identically in t, form an analytic subset of W, G. For $p \in W - G$ for all but a discrete set of values of t we have $rank\ (a_j^{(i)}(t, \varepsilon) = k$.

Without limiting generality, we assume U to be a polydisc. For any fixed $p_0 \in W - G$, there is a natural mapping from the space of solutions of the system to the space of solutions of the derived equation, which we can identify with the linear mapping

$$x_1(t_0, p_0), ..., x_n(t_0, p_0) \ \mapsto \ x_1(t_0, p_0), ..., x_1^{(k)}(t_0, p_0)$$

where t_0 is such that the leading coefficient does not vanish on t_0, p_0, given by

$$x_1^{(i)}(t_0, p_0) = (a_j^{(i)}(t_0, p_0))(x_j(t_0, p_0)).$$

Since the leading coefficient does not vanish at t_0, p_0, $rank\ (a_j^{(i)}(t, \varepsilon)) = k$, and the mapping is surjective. Hence $\forall p_0 \in W - G$ every solution of the derived equation comes from a solution of the system.

Remark: if the dependence is algebraic rather than holomorphic, the set G is algebraic.
Let us express the determinant of the i-th k-minor, the order of the derived equation being k, as $b_{i0}(p) + b_{i1}(p)t + ... + b_{ik_i}t^{k_i}$. We call the ideal generated by all $b_{ij}(p)$ the degeneracy ideal, and the corresponding variety the degeneracy variety.

We now make a small digression, and explain a basic difficulty which might occur in estimating the number of zeros of the first component of a solution of the system through the derived above equation for this component. In [2] one finds the following theorem, derived from growth estimates on the solutions:

Theorem. *Let*
$$a(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + ... + a_0(t)y = 0$$

have coefficients holomorphic in a region U containing a real segment K. Suppose that the modulus of all $a_i(t)$ is bounded in U by A, and that

$$max_{t \in K}|a(t)| \ \geq \ a.$$

Then the number of zeros on K of any holomorphic in U solution of the equation, real on the real axis, is not greater than
$$(A/a + n)^{\mu},$$

where μ depends only on the geometry of U and K.

The main obstacle in applying this theorem to our case lies in the fact that, as parameters vary, $max_{t \in K}|a(t, p)|$ (which depends on the parameters) *a priori* may become arbitrarily small - the situation of singular perturbations - and the estimates explode.

Our next goal is to show that the scalar differential equation we derived for x_1 cannot be in fact singularly perturbed, at least when one parameter is involved. This is somewhat unexpected, and we explain why. Consider the following very simple ε-dependent system:

$$\dot{x} = x + \varepsilon y \quad \dot{y} = x + y$$

One might expect that when $\varepsilon = 0$, the derived equation, which is of second order, will simply degenerate into the first order $\dot{x} = x$, implying that the derived equation is singularly perturbed. However, $\ddot{x} = x + \varepsilon y + \varepsilon(x + y) = (1 + \varepsilon)x + 2\varepsilon y$, and the derived equation is

$$\ddot{x} - 2\dot{x} + (1 - \varepsilon)x = 0.$$

This equation is not singularly perturbed. It has the property that for the exceptional value of the parameter, an additional solution appears which does not come from the original system, namely te^t. To explain why the singular perturbations cannot occur also in the general case, we proceed as follows. Until further notice the dimension of the parameter space is one, and the parameter will be denoted by ε.

Definition. We call a meromorphic function $f(t, \varepsilon)$ singularly perturbed at ε_0, if when expressed as a quotient of two holomorphic functions $f(t, \varepsilon) = u(t, \varepsilon)/v(t, \varepsilon)$, $u(t, \varepsilon)$ is divisible by a higher or equal power of $(\varepsilon - \varepsilon_0)$ than the one that divides $v(t, \varepsilon)$.

Definition. A (scalar) differential equation with meromorphic coefficients depending on one parameter, normalized so that the leading coefficient is equal to 1, will be called singularly perturbed if at least one of its coefficients is singularly perturbed at some point in the parameter space.

We make the following obvious claim.

Lemma 1. *Let us be given a differential equation with t, ε-dependent coefficients meromorphic in a domain U, singularly perturbed at ε_0. Then there is a point t_0, such that in the neighborhood of $(t_0, \varepsilon_0) \in U$, the equation takes the form*

$$(\varepsilon - \varepsilon_0)^s y^{(n)} + a_{n-1}(t, \varepsilon) y^{(n-1)} + \ldots + a_0(t, \varepsilon) y = 0,$$

$s > 0$, with coefficients holomorphic in the neighborhood of (t_0, ε_0), and at least one of them is not identically zero when $\varepsilon = \varepsilon_0$.

Theorem 1. *Let a differential equation of order n with coefficients which depend meromorphically on t, ε in some polydisc $U \times V$, be singularly perturbed for at least one parameter value ε_0, and let it have l solutions $y_1(t, \varepsilon), \ldots, y_l(t, \varepsilon)$ holomorphic in t and ε in that polydisc. Then for each fixed parameter value $\hat{\varepsilon}$ from V,*

$$dim_{\mathbb{C}} \ span_{\mathbb{C}} \left(y_1(t, \hat{\varepsilon}), \ldots, y_l(t, \hat{\varepsilon}) \right) \ \leq \ n - 1.$$

Proof. Using Lemma 1, we write the equation in the form given there in some smaller polydisc, and assume $t_0 = 0$, $\varepsilon_0 = 0$:

$$\varepsilon^s y^{(n)} + a_{n-1}(t, \varepsilon) y^{(n-1)} + \ldots + a_0(t, \varepsilon) y = 0,$$

with $a_k(0, 0) \neq 0$. We claim that if two holomorphic in t, ε solutions are such that they have equal $y^{(n-1)}(0, \varepsilon), \ldots, y^{(k+1)}(0, \varepsilon), y^{(k-1)}(0, \varepsilon), \ldots, y(0, \varepsilon)$, they are in fact identical. Indeed, otherwise we get that there exists a nonzero holomorphic in t, ε solution $y(t, \varepsilon)$ such that

$$y^{(n-1)}(0, \varepsilon) = \ldots = y^{(k+1)}(0, \varepsilon) = y^{(k-1)}(0, \varepsilon) = \ldots = y(0, \varepsilon) = 0.$$

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Since $y(t, \varepsilon)$ is not identically zero, $y^{(k)}(0, \varepsilon)$ cannot be identically zero. Thus $y^{(k)}(0, \varepsilon) = \varepsilon^h g(\varepsilon)$, $h \geq 0$, $g(0) \neq 0$. Assume $h > 0$. $y(t, 0)$ is the solution of the differential equation with ε set to zero. Since $a_k(0, 0) \neq 0$ and the initial conditions at $\varepsilon = 0$ are zero ($h > 0$), we have $y(t, 0) = 0$ identically. This can only happen if $y(t, \varepsilon) = \varepsilon y_1(t, \varepsilon)$, $y_1(t, \varepsilon)$ being also a holomorphic solution of the system with the same zero initial conditions except for $y_1^{(k)}(0, \varepsilon) = \varepsilon^{h-1} g(\varepsilon)$. We continue in the same way, until we get a holomorphic solution $y_h(t, \varepsilon) = y(t, \varepsilon)/\varepsilon^h$ with the initial conditions

$$y^{(n-1)}(0, \varepsilon) = \ldots = y^{(k+1)}(0, \varepsilon) = y^{(k-1)}(0, \varepsilon) = \ldots = y(0, \varepsilon) = 0,$$

and $y^{(k)}(o, \varepsilon) = g(\varepsilon)$, $g(0) \neq 0$. However substituting $y_h(t, \varepsilon)$ into the original equation and setting $t = 0$, we get:

$$\varepsilon^s y^{(n)}(0, \varepsilon) + a_k(0, \varepsilon) y^{(k)}(0, \varepsilon) = 0,$$

which is a contradiction $(a_k(0, 0), y^{(k)}(0, 0) \neq 0)$.

Thus in the smaller polydisc the holomorphic in t, ε solutions can be identified with sets of holomorphically dependent on ε sets of initial conditions $y^{(n-1)}(0, \varepsilon), \ldots, y^{(k+1)}(0, \varepsilon), y^{(k-1)}(0, \varepsilon), \ldots, y(0, \varepsilon)$, omitting $y^{(k)}(0, \varepsilon)$. Now we prove the theorem. Suppose we have holomorphic in $U \times V$ solutions of our singularly perturbed equation $y_1(t, \varepsilon), \ldots, y_l(t, \varepsilon)$. Over the field meromorphic in V functions there can be at most $n - 1$ linearly independent such solutions since the latter are identified with $\mathcal{O}(V)^{n-1}$. So we choose a maximal set S of linearly independent over $\mathcal{M}(V)$ solutions and express all the other as their linear combination over $\mathcal{M}(V)$. It follows that all the other solutions lie in the space spanned by the solutions from S for all but a discrete set of ε-s. Thus for almost all values of ε, the space of holomorphic in t, ε solutions is $n - 1$ dimensional, which implies that it is at most $n - 1$ dimensional for all values of ε by the semicontinuity of the dimension.

Theorem 2. *Suppose that we are given a linear differential system with coefficients depending on one parameter, holomorphic both in t and ε in a polydisc $U \times V$. Then the derived scalar differential equation for the variable x_1 cannot be singularly perturbed.*

Proof. Returning to notations of proof of Claim 1, we take a point $(t_0, \varepsilon_0) \in U - Z$, in which the leading coefficient of the derived equation is not zero, and $rank\ (a_j^{(i)}(t, \varepsilon) = k$.

For each m, there is a holomorphic in U solution of the system defined by the initial conditions (constant in ε):

$$x_1(t_0, \varepsilon) = 0 \ \ldots \ x_m(t_0, \varepsilon) = 1 \ \ldots \ x_n(t_0, \varepsilon) = 0.$$

We denote the 1-st component of this solution by $x_1^m(t, \varepsilon)$. By discussion from the proof of Claim 1, since $x_1^1(t, \varepsilon_0), \ldots, x_1^n(t, \varepsilon_0)$ span the space of solutions of the derived equation at ε_0, and since at this (generic) point the order of the derived equation is k, we have

$$dim_{\mathbb{C}}\ span_{\mathbb{C}}\ x_1^1(t, \varepsilon_0), \ldots, x_1^n(t, \varepsilon_0)\ =\ k.$$

However, if the derived equation is singularly perturbed even at only one point $\varepsilon_1 \in V$, from Theorem 1 it would follow that for every $\varepsilon_0 \in V$

$$dim_{\mathbb{C}}\ span_{\mathbb{C}}\ x_1^1(t, \varepsilon_0), \ldots, x_1^n(t, \varepsilon_0)\ \leq\ k - 1,$$

giving a contradiction. Thus the derived equation cannot be singularly perturbed.

We explain now our goal in what follows. As it was already mentioned, we can parameterize the problem of general linear system of order n with polynomial coefficients of degree d as:

$$\dot{x}_i = \sum_{j=1}^{n}\sum_{k=0}^{d} a_{ijk}t^k x_j \quad i=1,..,n$$

the parameters being a_{ijk}, thus making the system matrix to be polynomial in the time and parameters, with integer coefficients. Ideally we would like to bound the number of zeros which the first variable of this system might have, but the results available here are still not permitting such general bounds. Instead, we restrict ourselves to the case when the system matrix is polynomial in time and *one* additional parameter, then making several remarks about the general linear differential system with polynomial coefficients as above.

So let us have a linear differential system of order n, with coefficients being polynomials in t, ε of degree d, with integer coefficients of maximum modulus M. Let the order of the derived equation be k.

Lemma 2. *Let a nonsingularly perturbed rational function be written as*

$$\frac{a_0(\varepsilon) + a_1(\varepsilon)t + ... + a_k(\varepsilon)t^k}{b_0(\varepsilon) + b_1(\varepsilon)t + ... + b_l(\varepsilon)t^l}.$$

Then each a_j belongs to the ideal generated by $b_0, b_1, .., b_l$.

Proof. a_j must vanish at points where all $b_1, .., b_l$ vanish (otherwise the fraction would be singularly perturbed). Write $b_i(\varepsilon) = c_i(\varepsilon)b(\varepsilon)$, where $b(\varepsilon)$ is the common factor. Then $c_i(\varepsilon)$ have no common zero and therefore, since we are working over $\mathbb{C}$, $1 = h_1(\varepsilon)c_1(\varepsilon) + .. + h_l(\varepsilon)c_l(\varepsilon)$, for some polynomials $h_1, .., h_l$. But a_j must be divisible by $b(\varepsilon)$: otherwise we get singular perturbations. Thus:

$$a_j(\varepsilon) = d(\varepsilon)b(\varepsilon) = d(\varepsilon)b(\varepsilon)(h_1(\varepsilon)c_1(\varepsilon) + .. + h_l(\varepsilon)c_l(\varepsilon)) = d(\varepsilon)h_1(\varepsilon)b_1(\varepsilon) + .. + d(\varepsilon)h_l(\varepsilon)b_l(\varepsilon).$$

We derive now some computational estimates we will need.

Lemma 3. *Suppose we are given a non-singularly perturbed rational function of t, ε,*

$$\frac{a_0(\varepsilon) + a_1(\varepsilon)t + ... + a_k(\varepsilon)t^k}{b_0(\varepsilon) + b_1(\varepsilon)t + ... + b_l(\varepsilon)t^l},$$

with degrees of the numerator and the denominator being at most d, and the coefficients being integers of maximum modulus M (we mean by this possibly complex integers). Then one may write:

$$\frac{a_0(\varepsilon) + a_1(\varepsilon)t + ... + a_k(\varepsilon)t^k}{b_0(\varepsilon) + b_1(\varepsilon)t + ... + b_l(\varepsilon)t^l} = \frac{\sum_{i=1}^{k} t^i \sum_{j=1}^{l} h_{ij}(\varepsilon)b_j(\varepsilon)}{b_0(\varepsilon) + b_1(\varepsilon)t + ... + b_l(\varepsilon)t^l},$$

with $h_{ij}(\varepsilon)$ being polynomials of degree at most $2d-1$ with integer coefficients of maximum modulus $(eM)^{Cd^3}$ for some constant C.

Proof. From Lemma 2, $a_i = \sum h_{ij}b_j$. By [5], since $\forall i\ deg(a_i) \le d$, $\forall j\ deg(b_j) \le d$, we have that $a_i = \sum h_{ij}b_j$, $deg(h_{ij}) \le 2d-1$. Observe that this gives us an integer coefficient linear system of dimension at most $2d(d+1)$, maximum modulus of the coefficients not greater than M. By the integrality trick this gives us the estimate $(2d(d+1))!M^{2d(d+1)} \le (eM)^{Cd^3}$ on the coefficients of

the polynomials h_{ij}.

Lemma 4. *In the setting of Lemma 3, suppose that $|\varepsilon|$ is bounded by E. Then the above fraction can always be written as*

$$\frac{\sum_{i=1}^{d} c_i t^i}{\sum_{i=1}^{d} d_i t^i}$$

with $c_i \leq (eM)^{Cd^3}(E^{2d-1}+1)$ for some constant C , with some d_i equal to 1 and all $|d_i|$ being not greater than 1.

Proof. For each ε in the disc, take the b_i having the maximum modulus, and divide by it both the numerator and the denominator: use Lemma 3 (the constant here is different from the one in Lemma 3).

Lemma 5. *Suppose we are given a non-singularly perturbed differential equation of order k*

$$y^{(k)} + \frac{a_{k-1}(t,\varepsilon)}{b(t,\varepsilon)} y^{(k-1)} + ... + \frac{a_0(t,\varepsilon)}{b(t,\varepsilon)} y = 0,$$

all $a_i(t,\varepsilon)$ and $b(t,\varepsilon)$ being polynomials of degree at most d, having integer coefficients of maximum modulus M.
Let $[-R/2, R/2]$ be a segment on the real axis containing $[-1, 1]$. Fix any real ε of modulus less than E. Let f be a holomorphic solution of the equation for this parameter value, real on the real axis. Then the number of zeros f can have on $[-R/2, R/2]$, is bounded by

$$\left((Me)^{Cd^3}(E^{2d}+1)R^{d+1} + k \right)^{\sigma},$$

with σ and C being some constants.

Proof. By Lemma 4, we can write the equation in the form

$$(t^s + \sum_{i \neq s} d_i t^i) y^{(k)} + (c_{k-1\ 0} + .. + c_{k-1\ d} t^d) y^{(k-1)} + ... + (c_{00} + ... + c_{0d} t^d) y = 0.$$

with $|c_{ij}|$ being bounded by the bound from Lemma 2, and all $|d_i| \leq 1$. If we could find now some point in $[-R/2, R/2]$ where $t^s + \sum_{i \neq s} d_i t^i$ has some value bounded away from zero, we are done. This because then we may just apply the Theorem cited from [2].
So we write

$$|t^s + \sum_{i \neq s} d_i t^i| \geq |t^s + d_{s-1} t^{s-1} + ... + d_0| - |t^{s+1}(d_{s+1} + d_{s+2} t + ..)|$$

and notice that by Cartan lemma ([2]), one may delete disks of total diameter h from $\mathbb{C}$, such that on the rest of $\mathbb{C}$ the modulus of the monic polynomial $t^s + d_{s-1} t^{s-1} + ... + d_0$ is $\geq (h/4e)^s$. Thus for each $h > 0$ there exists a point t_h of modulus at most $h/2$, at which the value of the polynomial is $\geq (h/4e)^s$. But then:

$$|t_h^s + \sum_{i \neq s} d_i t_h^i| \geq |t_h^s + d_{s-1} t_h^{s-1} + ... + d_0| - |t_h^{s+1}(d_{s+1} + d_{s+2} t_h + ..)| \geq (h/4e)^s - h^{s+1}(d-s).$$

Taking for example $h = 1/(2(4e)^s(d-s))$ we get that there exists a point on the unit segment at which $|t^s + \sum_{i \neq s} d_i t^i|$ is bounded from below by

$$\frac{1}{(4e)^{s^2+s}2^{s+1}(d-s)^s} \geq \frac{1}{(4e)^{s^2+s}2^{s+1}d^s}$$

As s can be from 0 to $d-1$ (it can be shown that the case $s = d$ is also covered by the obtained estimate), we are led to the conclusion that there will be a point in $[-R/2, R/2]$ where

$$|t^s + \sum_{i \neq s} d_i t^i| \geq \frac{1}{(4e)^{d^2+d}2^{d+1}d^d}.$$

In turn, $|\sum_{j=0}^d c_{ij}t^j|$ cannot, when ε varies over disk of radius E and t over D_R, be larger than

$$\sum_{j=0}^d R^j \cdot (eM)^{Cd^3}(E^{2d-1}+1) \quad \leq \quad (eM)^{Cd^3}(E^{2d-1}+1)R^{d+1}.$$

By Theorem cited from [2], one deduces that for any given real $|\varepsilon| \leq E$, any holomorphic in t solution of the given differential equation, real on the real axis, may have not more than

$$\left(\frac{(eM)^{Cd^3}(E^{2d-1}+1)R^{d+1}}{1/(4e)^{d^2+d}2^{d+1}d^d} + k\right)^\sigma$$

zeros on $[-R/2, R/2]$, σ not depending on R because of properties of $\sigma(K, U)$ from the Theorem. Simplifying, one gets the bound we stated.

To estimate the number of zeros a solution (more precisely, its x_1 component) of the initial linear system

$$\dot{x} = A(t, \varepsilon)x$$

might have, we now have only to bound the maximum modulus of the (integer) coefficients of the derived equation. We will do the computation for an arbitrary number of parameters; to emphasize this we shall write p instead of ε. To remind the notations, we denote the dimension of the system by n, the degree of the polynomial coefficients by d, the maximum modulus of the coefficients of those polynomials by M, the number of parameters by q, and the order of the derived equation for x_1 by k.

Lemma 6. *The following recurrent formulas hold for $a^{(i)}$, defined in the beginning of this text:*

$$a^{(i)} = \dot{a}^{(i-1)} + A(t, p)^T a^{(i-1)}.$$

Proof. Immediate.

The following is a computation:

Lemma 7. *Let P_1, P_2 be two polynomials of degrees d_1, d_2 in s variables, with the modulus of their coefficients being not larger than M_1, M_2, respectively. Then the degree of $P_1 P_2$ is $d_1 + d_2$ and the coefficients are not larger in absolute value than $(1 + min(d_1, d_2))^s M_1 M_2$.*

Lemma 8. *The degrees and the maximum modulus of coefficients of the entries of $a^{(i)}$ are di*

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and $n^i(d + (d+1)^{q+1}M)^i$, respectively.

Proof. By writing recursive relations.

We now compute the maximum modulus and the degrees for the coefficients of the derived equation.

Lemma 9. Let

$$y^{(k)} + \frac{\alpha_{k-1}(t,p)}{\beta(t,p)}y^{(k-1)} + ... + \frac{\alpha_0(t,p)}{\beta(t,p)}y = 0$$

be the equation derived for x_1 from the original system. Then the degrees of both α_i and β are bounded by $k(k+1)d/2$, and the maximum modulus for the (integer) coefficients of those polynomials is not larger than $(eM(d+1))^{C(q+1)n^3}$ for some constant C.

Proof. Again by recursive relations.

Theorem 2. *Let the initial linear differential system have as coefficients polynomials in t, ε (ε one dimensional) with integer coefficients of maximum modulus M, and let $[-R/2, R/2]$ be a segment on the real axis containing $[-1, 1]$. Fix any real ε of modulus less than E. Let f be a holomorphic solution of the equation for this parameter value, real on the real axis. Then the number of zeros f can have on $[-R/2, R/2]$, is bounded by*

$$\left((Me)^{Cn^9 d^4}(E^{n(n+1)d} + 1)R^{n(n+1)d/2+1} + n \right)^\sigma,$$

for some constants C and σ.

Proof. Just substitute the bounds stated in Lemma 9 with $q = 1$ into Lemma 5 and simplify the expression.

Of course, the concrete form of the bound given above is irrelevant; what is important is the rate of its growth, which is simple exponential in n, d compared with the tower of exponents which can be derived from [1] in this situation.

We now make some remarks on the general case when many parameters are allowed. First, it is clear that the absence of singular perturbations in the case of dependence on one parameter implies absence of singular perturbations along any given holomorphic curve in the parameter space, and the following is true.

Lemma 2a. *Let a rational function dependent on several parameters be written as*

$$\frac{a_0(p) + a_1(p)t + ... + a_k(p)t^k}{b_0(p) + b_1(p)t + ... + b_l(p)t^l},$$

and let it be such that it is not singularly perturbed along any line in the parameter space. Then each a_j vanishes on the variety where all $b_0, b_1, .., b_l$ vanish.

We cannot in general have the conclusion of Lemma 2 as the example of the nonsingularly perturbed fraction

$$\frac{ab}{a^2 + b^2 t}$$

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shows. That is, in the derived equation for system with polynomial coefficients depending on many parameters we can only guarantee that the coefficients in the numerator vanish on the degeneracy variety, but not that they in fact belong to the degeneracy ideal. However, if by some additional reasoning for, say, the general linear system with polynomial coefficients

$$\dot{x}_i = \sum_{j=1}^{n} \sum_{k=0}^{d} a_{ijk} t^k x_j \quad i = 1, .., n,$$

it will turn out to be true, then the bound, though drastically grows because of the double exponential in $n^2(d+1)$ bounds in the effective Nullstellensatz (see [5]), has the asymptotic form (in terms of Theorem 2)

$$e^{e^{e^{P(n,d)}}} (ER)^{Cn^2 d},$$

where P is a polynomial. This is still a considerable improvement compared to [1] (of course if the assumption is valid).

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